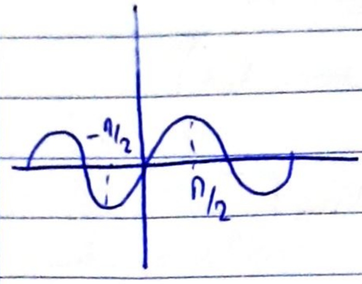
06/05/2019

(Απειροστικός 2
Σαρόγλου)

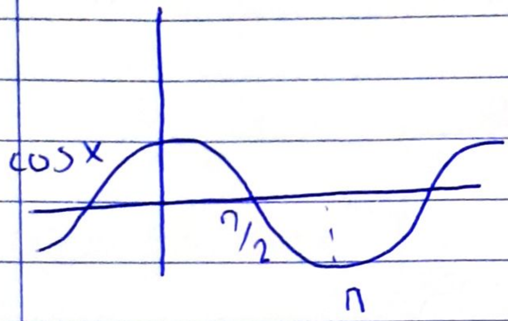
## Αντίστροφες Τριγωνομετρικές Συναρτήσεις




$\sin^{-1}: [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

||

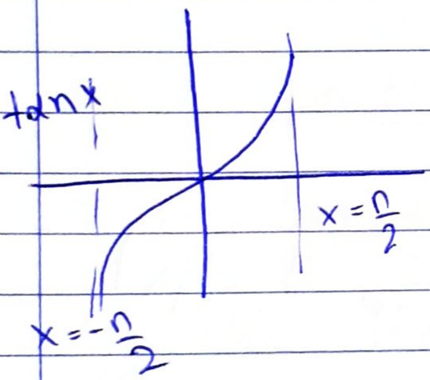
Arcsin




$\cos^{-1} [-1,1] \to [0,\pi]$

||

Arccos




$\tan^{-1}: \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

||

Arctan

**ΘΕΩΡΗΜΑ** Έστω $f: [\alpha,b] \to [c,d]$ "1-1", επί, παραγωγίσιμη και $f'(x) \neq 0$, $\forall x \in [\alpha,b]$. Τότε $f^{-1}$ είναι παραγ. στο $[c,d]$ και ισχύει $(f^{-1})'(y) = \dfrac{1}{f'(x)}$, όπου $y = f(x)$.

Arcsin : $[-1,1] \to [-\pi/2, \pi/2]$, Arccos $[-1,1] \to [0,\pi]$

Arctan : $\mathbb{R} \to (-\pi/2, \pi/2)$, Arccty : $\mathbb{R} \to [-\pi/2, \pi/2] \setminus \{0\}$

$(\text{Arcsin}\, y)' = (\sin^{-1} y)' = \dfrac{1}{(\sin x)'} = \dfrac{1}{\cos x}$ $(y = \sin x)$

$\xrightarrow[x \in (-\frac{\pi}{2}, \frac{\pi}{2})]{\cos x \geq 0} \dfrac{1}{\sqrt{1-\sin^2 x}} = \dfrac{1}{\sqrt{1-y^2}}$, $y \in (-1,1)$

$$(\text{Arccos}y)' = (\cos^{-1} y)' = \frac{1}{(\cos x)'} = \frac{-1}{\sin x} \quad (y = \cos x)$$

$$\xrightarrow[x \in (0,\pi)]{\sin x > 0} \frac{-1}{\sqrt{1-\cos^2 x}} = -\frac{1}{\sqrt{1-y^2}}, \ y \in (-1,1)$$

$$(\text{Arctan}y)' = (\tan^{-1} y) = \frac{1}{(\tan x)'} \overset{(y=\tan x)}{=} \cos^2 x = \frac{1}{\tan^2 x + 1}$$

$$\boxed{\begin{gathered}(\tan x') = \frac{1}{\cos^2 x} \\ \tan^2 x + 1 \\ = \frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}\end{gathered}}$$

$$= \frac{1}{y^2 + 1}$$

το $y \in \mathbb{R}$

$$(\text{Arctg}y)' = (ctg^{-1} y)' = \frac{1}{(ctg\, x)'} \quad (y = ctg x)$$

$$\boxed{\begin{gathered}(ctg x') = -\frac{1}{\sin^2 x} \\ ctg^2 x + 1 = \frac{1}{\sin^2 x}\end{gathered}}$$

$$= \sin^2 x = \frac{-1}{(ctg x)^2 + 1} = \frac{-1}{y^2+1}$$

το $y \in \mathbb{R}$

**Πόρισμα**

$$\int \frac{1}{\sqrt{1-y^2}}\, dy = \text{Arcsin}y + C, \ y \in (-1,1)$$

$$\int \frac{1}{\sqrt{y^2+1}}\, dy = \text{Arctan}y + C, y \in \mathbb{R}$$

Παράδειγμα: $\int \frac{\operatorname{Arctan} x}{1+x^2}\,dx \xrightarrow[du=\frac{1}{1+x^2}dx]{u=\operatorname{Arctan} x} \int u\,du = \frac{u^2}{2}+C$

$= \frac{(\operatorname{Arctan} x)^2}{2} + C \quad , x \in \mathbb{R}$

## 1ο ΘΕΩΡΗΜΑ ΑΝΤΙΚΑΤΑΣΤΑΣΗΣ

Έστω $\varphi: [\alpha, b] \to \mathbb{R}$ παραγωγίσιμη, $\varphi'$ ολοκληρώσιμη $f: \varphi([\alpha, b]) \to \mathbb{R}$ συνεχής τότε $\int_\alpha^b f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(b)} f(s)\,ds$

## 2ο ΘΕΩΡΗΜΑ ΑΝΤΙΚΑΤΑΣΤΑΣΗΣ

Έστω $\varphi: [\alpha, b] \to \mathbb{R}$, συνεχώς παραγ. με $\varphi'(x) \neq 0$ $\forall x \in [\alpha, b]$ $f: (\varphi[\alpha, b]) \to \mathbb{R}$ συνεχής τότε $\int_\alpha^b f(\varphi(t))\,dt =$

$= \int_{\varphi(\alpha)}^{\varphi(b)} f(s)(\varphi^{-1})'(s)\,ds$

## ΤΡΙΓΩΝΟΜΕΤΡΙΚΑ ΟΛΟΚΛΗΡΩΜΑΤΑ

1) $\int \sin^m x \cos^n x\,dx$, $m, n \in \mathbb{N} \cup \{0\}$ τουλάχιστον ένας εκ των $m$ κ' $n$ περιττός

α) $n$ περιττός: $n = 2k+1$ $\int \sin^m x \cos^n x\,dx = \int \sin^m x (\cos^2 x)^k \cos x\,dx$

$= \int \sin^m x\,(1-\sin^2 x)(\sin x)'\,dx$

Αν το ολοκλήρωμα είναι ορισμένο $f(x) = u^m(1-u^2)$ $\varphi(x) = \sin x$

$\int_\alpha^b f(\varphi(x))\varphi'(x)\,dx \xrightarrow{1^{ο}\ \Theta.A} \int_{\sin\alpha}^{\sin b} u^m(1-u^2)\,du$

b) Αν m περιττός: $m = 2k+1$ $\int \sin^m x \cos^n x \, dx =$

$= \int (\sin^2 x)^k \cos^n x \sin x \, dx$

$= -\int (1-\cos^2 x)^k \cos^n x (\cos x)' dx \underline{\underline{u=\cos x}}$

$-\int (1-u^2)^k u^n du$

Παράδειγμα: $\int_0^{\frac{3\pi}{2}} \sin^5 x \cos^2 x \, dx = \int_0^{\frac{3\pi}{2}} (\sin^2 x)^2 \cos^2 x \sin x \, dx$

$= -\int_0^{\frac{3\pi}{2}} (1-\cos^2 x)^2 \cos^2 x (\cos x)' dx \underline{\underline{1^ο\ Θ.\ Ανεικ.}}$

$u = \cos x$

$\cos 0 = 1$

$\cos \frac{3\pi}{2} = 0$

$-\int_1^0 (1-u^2)^2 u^2 du = \int_0^1 (1-u^2)^2 u^2 du =$

$= \int_0^1 (1+u^4-2u^2) u^2 du = \int_0^1 (u^2+u^6-2u^4) du$

$= \left[ \frac{u^3}{3} + \frac{u^7}{7} - 2\frac{u^5}{5} \right]_0^1 = \frac{1}{3} + \frac{1}{7} - \frac{2}{5}$

2) $\int \sin^{2k} x \cos^{2\ell} x \, dx$, $k, \ell \in \mathbb{N} \cup \{0\}$

Χρησιμοποιούμε: $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Παράδειγμα: $\int_0^{\pi} \sin^4 x \, dx = \int_0^{\pi} \left[ \frac{1}{2}(1-\cos 2x) \right]^2 dx$

$$= \frac{1}{4}\int_0^{\pi} (1 - 2\cos 2x + \cos^2(2x))\,dx$$

$$= \frac{1}{4}[x - \sin 2x]_0^{\pi} + \frac{1}{4}\int_0^{\pi} \cos^2(2x)\,dx$$

$$= \frac{\pi}{4} + \frac{1}{4}\int_0^{\pi} \frac{1}{2}(1+\cos 4x)\,dx = \frac{\pi}{4} + \frac{1}{4}\left[\frac{1}{2}\left(x + \frac{\sin 4x}{4}\right)\right]_0^{\pi} =$$

$$= \frac{\pi}{4} + \frac{\pi}{8} = \frac{3\pi}{8}$$

3) $\int \sin(mx)\cos(nx)\,dx$, $\int \sin(mx)\sin(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$, $m,n \in \mathbb{Z}$

Χρησιμοποιούμε $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Παράδειγμα $\int \sin^4 x \cos 5x\,dx = \frac{1}{2}\int (\sin(-x) + \sin(9x))\,dx$

$$= 1/2\left(+\cos x - \frac{\cos(9x)}{9}\right) + C$$

4) $\int \frac{\tan^m x}{\cos^n x}\,dx \quad n = 2k$

$$\int \tan^m x \left(\frac{1}{\cos^2 x}\right)^{k-1} \frac{1}{\cos^2 x}\,dx = \int \tan^m x (\tan^2 x + 1)^{k-1} (\tan x)'\,dx$$

$$\overset{u=\tan x}{=\!=} \int u^m (u^2+1)^{k-1}\,du$$

ΠΑΡΑΔΕΙΓΜΑ: $\int \frac{\tan^6 x}{\cos^4 x}\,dx = \int \tan^6 x \frac{1}{\cos^2 x}\,dx$

$= \tan^6 x\,(\tan^2 x + 1)(\tan x)'\,dx \;\; \underline{\underline{u=\tan x}}$

$\int u^6(u^2+1)\,du = \int (u^8+u^6)\,du = \frac{u^9}{9} + \frac{u^7}{7} + c$

$\int \frac{\tan^m x}{\cos^n x}\,dx \quad , m = 2k+1$

$\int \frac{(\tan^2 x)^k}{\cos^n x}\frac{\tan x}{\cos x}\,dx = \int \frac{\left(\frac{1}{\cos^2 x}-1\right)^k}{\cos^{n-1} x}\frac{\tan x}{\cos x}\,dx$

$= \int \left(\frac{1}{\cos^2 x} - 1\right)^k \left(\frac{1}{\cos x}\right)^{n-1} \frac{\tan x}{\cos x}\,dx$

$\left(\frac{1}{\cos x}\right)' = \frac{-(\cos x)'}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\tan x}{\cos x}$

$\underline{\underline{u = 1/\cos x}} \int (u^2-1)^k u^{n-1}\,du$

ΠΑΡΑΔΕΙΓΜΑ $\int \frac{\tan^5 x}{\cos^2 x}\,dx = \int \frac{(\tan^2 x)^2}{\cos^6 x}\frac{\tan x}{\cos x}\,dx$

$= \left(\frac{1}{\cos^2 x} - 1\right)^2 \left(\frac{1}{\cos^2 x}\right)\left(\frac{1}{\cos x}\right)' dx$ *

$= \frac{(1/\cos x)^{11}}{11} - 2\frac{(1/\cos x)^9}{9} + \frac{(1/\cos x)^7}{7} + c$

* $u = 1/\cos x \int (u^2-1)^2 u^6\,du = \int (u^4 - 2u^2 + 1)u^6\,du =$
$\int (u^{10} - 2u^8 + u^6)\,du = \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + c$

$\sqrt{a^2-x^2}$ $\quad x = a\sin\vartheta \quad , \quad -\pi/2 < \vartheta < \pi/2$

$\sqrt{a^2+x^2}$ $\quad x = a\tan\vartheta \quad , \quad -\pi/2 < \vartheta < \pi/2$

$\sqrt{x^2-\alpha^2}$ $\quad x = \dfrac{\alpha}{\cos\vartheta} \quad , \quad 0 \le \vartheta < \dfrac{\pi}{2}$ ή

$\pi \le \vartheta < \dfrac{3\pi}{2}$

ΠΑΡΑΔΕΙΓΜΑΤΑ 1. $\displaystyle\int_1^{\sqrt{3}} \underbrace{\frac{1}{x^2\sqrt{x^2+4}}}_{f(x)} dx$

$f(\mathrm{Arctan}(\tan x))$

$f\circ \mathrm{Arctan}$

$\cancel{\quad} \; f(x) = g\left(\mathrm{Artan}\,\dfrac{x}{\alpha}\right)$

$f(\varphi(t)) = g(t)$

$f(\alpha\tan\vartheta) = g(\vartheta) \qquad \varphi(\vartheta) = a\tan\vartheta$

$\quad ||$

$g\left(\mathrm{Arctan}\,\dfrac{\alpha\tan\vartheta}{\alpha}\right) \qquad f(x) = g\left(\mathrm{Arctan}\dfrac{x}{\alpha}\right) = g\left(\mathrm{Arctan}\dfrac{x}{2}\right)$

2Θ. Αντικατάσταση

$x=2 \Leftrightarrow \vartheta = \frac{\pi}{4}$

$x = 2\sqrt{3} \Leftrightarrow \vartheta = \frac{\pi}{3}$

$(2\tan\vartheta)' \neq 0$

$\forall \vartheta \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$

$$\int_{\pi/4}^{\pi/3} \frac{1}{4\tan^2\vartheta}\,\frac{2}{\cos^2\vartheta}\,d\vartheta = \int_{\pi/4}^{\pi/3} \frac{1}{4\tan^2\vartheta\cos\vartheta}\,d\vartheta = \int_{\pi/4}^{\pi/3} \frac{1}{4}\,\frac{\cos\vartheta}{\sin^2\vartheta}\,d\vartheta$$

$$\boxed{\begin{aligned} &x^2 = 4\tan^2\vartheta \\ &\sqrt{x^2+4} = \sqrt{4(1+\tan^2\vartheta)} \\ &= 2\sqrt{\frac{1}{\cos^2\vartheta}} = \frac{2}{\cos\vartheta} \\ &dx = \frac{2}{\cos^2\vartheta}\,d\vartheta \end{aligned}}$$